

The Generalized Uncertainty Principle from Quantum Geometry

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Abstract

The generalized uncertainty principle of string theory is derived in the framework of Quantum Geometry by taking into account the existence of an upper limit on the acceleration of massive particles.

PACS: 11.17.+y; 04.62.+v

Keywords: Quantum Geometry, Maximal Acceleration, String theory.

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1 Introduction

The problem of reconciling Quantum Mechanics (QM) with General Relativity is one of the task of modern theoretical physics which, until now, has not yet found a consistent and satisfactory solution. The difficulty arises since general relativity deals with the events which define the world-lines of particles, while QM does not allow the definition of trajectory; in fact the determination of the position of a quantum particle involves a measurement which introduces an uncertainty into its momentum (Wigner, 1957; Saleker, 1958; Feynman, 1965).

These conceptual difficulties have their origin, as argued in Ref. (Candelas, 1983; Donoghue, 1984, 1985), in the violation, at quantum level, of the weak principle of equivalence on which general relativity is based. Such a problem becomes more involved in the formulation of quantum theory of gravity, owing to the non-renormalizability of general relativity when one quantizes it as a local Quantum Field Theory (QFT) (Birrel, 1982).

Nevertheless, one of the most interesting consequences of this unification is that in quantum gravity there exists a minimal observable distance on the order of the Planck distance, $l_P = \sqrt{G\hbar/c^3} \sim 10^{-33}\text{cm}$, where G is the Newton constant. The existence of such a fundamental length is a dynamical phenomenon due to the fact that, at Planck scales, there are *fluctuations* of the background metric, *i.e.* a limit of the order of Planck length appears when quantum fluctuations of the gravitational field are taken into account.

In absence of a theory of quantum gravity, one tries to analyze quantum aspects of gravity retaining the gravitational field as a classical background, described by general relativity, and interacting with matter field. This *semiclassical approximation* leads to QFT and QM in curved space-time and may be considered as a preliminary step towards the complete quantum theory of gravity. In other words, we take into account a theory where geometry is classically defined while the source of Einstein equations is an effective stress-energy tensor where contributions of matter quantum fields, gravity self-interactions, and quantum matter-gravity interactions appear (Birrel, 1982).

Besides, the canonical commutation relations between the momentum operator p^ν and position operator x^μ , which in Minkowski space-time are $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}$, in a curved space-time with metric $g_{\mu\nu}$ can be generalized as

$$[x^\mu, p^\nu] = i\hbar g^{\mu\nu}(x). \quad (1)$$

Eq. (1) contains gravitational effects of a particle in first quantization scheme. Its validity is confined to curved space-time asymptotically flat so that the tensor metric can be decomposed as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the (local) perturbation to the flat background (Ashtekar, 1990). We note that the usual commutation relations between position and momentum operators in Minkowsky space-time are obtained by using the veirbein formalism, *i.e.* by projecting the commutator and the metric tensor on the tangent space.

As it is well known, a theory containing a fundamental length on the order of l_P (which can be related to the extension of particles) is string theory. It provides a consistent theory of quantum gravity and allows to avoid the above mentioned difficulties. In fact, unlike point particle theories, the existence of a fundamental length plays the role of natural cut-off. In such a way the, ultraviolet divergencies are avoided without appealing to the renormalization and regularization schemes (Green, 1987).

Besides, by studying string collisions at planckian energies and through a renormalization group type analysis (Veneziano, 199; Amati, 1987, 1988, 1989, 1990; Gross, 1987, 1988; Konishi, 1990; Guida, 1991; Yonega, 1989), the emergence of a minimal observable distance yields to the generalized uncertainty principle

$$\Delta x \geq \frac{\hbar}{2\Delta p} + \frac{\alpha}{c^3} G \Delta p. \quad (2)$$

Here, α is a constant. At energy much below the Planck mass, $m_P = \sqrt{\hbar c/G} \sim 10^{19} \text{GeV}/c^2$, the extra term in Eq. (2) is irrelevant and the Heisenberg relation is recovered, while, as we approach the Plack energy, this term becomes relevant and, as said, it is related to the minimal observable length.

The purpose of this paper is to recover the generalized uncertainty principle, Eq. (2), in the framework of Quantum Geometry theory (Caianiello, 1979, 1980a, 1980b, 1992). It tries to incorporate quantum aspects into space-time geometry so that one-particle QM may acquire a geometric interpretation. Its formulation is based on the fact that the position and momentum operators are represented as covariant derivatives with an appropriate connection in the eight-dimensional manifold and the quantization is geometrically interpreted as curvature of phase space.

A consequence of this geometric approach is the existence of maximal acceleration defined as the upper limit to the proper acceleration \mathcal{A} experienced by massive particles along their worldlines (Caianiello, 1981, 1982, 1984). It can be interpreted as mass-dependent, $\mathcal{A}_m = 2mc^3/\hbar$ (m is the mass of particle), or as an universal constant, $\mathcal{A} = m_P c^3/\hbar$ (m_P is the Planck mass). Since the regime of validity of (2) is at Planck scales, in order to derive it from quantum geometry, we will consider maximal acceleration depending on Planck mass.

The existence of a maximal acceleration has several implications for relativistic kinematics (Scarpetta, 1984), energy spectrum of a uniformly accelerated particle (Caianiello, 1990a), Schwarzschild horizon (Gasperini, 1989), expansion of the very early universe (Caianiello, 1991), tunneling from *nothing* (Capozziello, 1993; Caianiello, 1994), and mass of the Higgs boson (Kuwata, 1996). It also makes the metric observer-dependent, as conjectured by Gibbons and Hawking (Gibbons, 1977) and leads, in a natural way, to the hadronic confinement (Caianiello, 1988). Besides, the regularizing properties of the maximal acceleration has been recently analyzed in Ref. (Feoli, 1999), and its applications in the framework of string theory have been studied in Refs. (Feoli, 1993; McGuigan, 1994).

Moreover, concrete experimental tests of the consequence of the maximal acceleration have been proposed in Refs. (Caianiello, 1990; Papini, 1995a; Lambiase, 1998).

Limiting values for the acceleration were also derived by several authors on different grounds and applied to many branches of physics (Brandt, 1983, 1984, 1989; Das, 1980; Frolov, 1991; Papini, 1992, 1995b; Pati, 1992; Sanchez, 1993; Toller, 1988, 1990, 1991; Vigier, 1991; Wood, 1989, 1992).

The paper is organized as follows. In Section 2, we shortly discuss quantum geometry theory, recalling only the main topics used in this paper. Section 3 is devoted to derive the generalized uncertainty principle from quantum geometry. Conclusions are discussed in Section 4.

2 Quantum Geometry Theory

Quantum geometry includes the effects of the maximal acceleration on dynamics of particles in enlarging the space-time manifold to an eight-dimensional space-time tangent bundle TM, *i.e.* $M_8 = V_4 \otimes TV_4$, where V_4 is the background space-time equipped with metric $g_{\mu\nu}$. In this way, the invariant line element defined in M_8 is generalized as

$$d\tilde{s}^2 = g_{AB}dX^A dX^B, \quad A, B = 1, \dots, 8, \quad (3)$$

where the coordinates of M_8 are

$$X^A = \left(x^\mu; \frac{c^2}{\mathcal{A}} \frac{dx^\mu}{ds} \right), \quad \mu = 1, \dots, 4. \quad (4)$$

ds is the usual infinitesimal element line, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, \mathcal{A} is the maximal acceleration and

$$g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu}. \quad (5)$$

From Eq. (5), it follows that the generalized line element (3) can be written as

$$d\tilde{s}^2 = g_{\mu\nu}(dx^\mu dx^\nu + \frac{c^4}{\mathcal{A}^2} d\dot{x}^\mu d\dot{x}^\nu). \quad (6)$$

An embedding procedure can be developed (Caianiello, 1990b) in order to find the effective space-time geometry where a particle moves when the constraint of the maximal acceleration is present. In fact, if we find the parametric equations that relate the velocity field \dot{x}^μ to the first four coordinates x^μ , we can calculate the effective four dimensional metric $\tilde{g}_{\mu\nu}$ induced on the hypersurface locally embedded in M_8 . For a particle of mass m accelerating along its worldline, Eq. (6) implies that it behaves dynamically as if it is embedded in a space-time with the metric

$$d\tilde{s}^2 = \left(1 + c^4 \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{\mathcal{A}^2} \right) ds^2, \quad (7)$$

or, in terms of metric tensor

$$\tilde{g}_{\mu\nu} = \left(1 + c^4 \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{\mathcal{A}^2}\right) g_{\mu\nu}, \quad (8)$$

that depends on the squared length of the (spacelike) four-acceleration, $|\ddot{x}|^2 = g_{\sigma\rho} \ddot{x}^\sigma \ddot{x}^\rho$. Particularly interesting is the case in the absence of gravity, $g_{\mu\nu} = \eta_{\mu\nu}$ which corresponds to a flat background. In this case, any accelerating particle experiences a gravitational field given by

$$\tilde{g}_{\mu\nu} = \left(1 + c^4 \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{\mathcal{A}^2}\right) \eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (9)$$

where $h_{\mu\nu} = c^4(\ddot{x}^\sigma \ddot{x}_\sigma / \mathcal{A}^2) \eta_{\mu\nu}$ is the quantum (local) perturbation to the Minkowskian metric. From Eq. (9) it follows that

$$\tilde{g}^{\mu\nu} \sim \left(1 - c^4 \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{\mathcal{A}^2}\right) \eta^{\mu\nu}. \quad (10)$$

Nevertheless, we stress that this curvature is not induced by matter through conventional Einstein equation; it is due to the motion in momentum space and vanishes in the limit $\hbar \rightarrow 0$. Thus, it represents a quantum correction to the given background geometry, that, henceforth, we will assume flat.

3 Generalized Uncertainty Principle

Let us now derive the generalized uncertainty principle (2) starting from relation (1), where the tensor metric is induced by the acceleration of a massive particle in a high energy scattering process.

According to the hypothesis that microscopic space-time should be regarded as a four-dimensional hypersurface locally embedded in the larger height-dimensional manifold, as discussed in the previous section, accelerated particles can be associated to four-dimensional hypersurfaces whose curvature is, in general, non vanishing. At this semiclassical level, the effective space-time geometry experimented by interacting particles is curved.

Inserting (9) into (1), one gets

$$[x^\mu, p^\nu] = i\hbar \left(1 + c^4 \frac{(\ddot{x}^\sigma \ddot{x}_\sigma)_m}{\mathcal{A}^2}\right)^{-1} \eta^{\mu\nu}. \quad (11)$$

The right-hand side is understood as a c -function. The term $(\ddot{x}^\sigma \ddot{x}_\sigma)_m$ is the mean value of the squared length of the four-acceleration which takes into account the quantum fluctuation of the metric.

Since $\ddot{x}^\mu = (1/mc)\delta p^\mu / \delta s$, δp^μ is the transferred momentum, it follows that

$$(\ddot{x}^\sigma \ddot{x}_\sigma)_m \simeq \frac{1}{m^2 c^2 \delta s^2} \left[\frac{p^i p^j}{|\vec{p}|^2} - \delta^{ij} \right] (\delta p^i \delta p^j)_m, \quad i, j = 1, 2, 3, \quad (12)$$

where the high energy limit $E \gg m$ has been used. Due to the average on the product of transferred momenta, one can assume

$$(\delta p^i \delta p^j)_m \sim \Delta p^2 \delta^{ij}, \quad (13)$$

then Eq. (12) reads as

$$(\ddot{x}^\sigma \ddot{x}_\sigma)_m \sim -2 \frac{\Delta p^2}{m^2 c^2 \delta s^2}. \quad (14)$$

Δp is the transferred momentum along the x -direction.

As it is well known, two non-commuting operators A and B defined in a Hilbert space, for any given state, satisfy the uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2} | \langle [A, B] \rangle |.$$

If $A = x^\mu$ and $B = p^\nu$, Eqs. (10) and (11) allow to write

$$\Delta x^\mu \Delta p^\nu \geq \frac{\hbar}{2} |\eta^{\mu\nu}| |1 - c^4 \frac{(\ddot{x}^\sigma \ddot{x}_\sigma)_m}{\mathcal{A}^2}|. \quad (15)$$

From Eq. (14) and for $\mu = \nu = 1$, one obtains

$$\Delta x \Delta p \geq \frac{\hbar}{2} + \frac{\hbar c^2}{m^2 \mathcal{A}^2 \delta s^2} \Delta p^2. \quad (16)$$

For $\mathcal{A} = m_P c^3 / \hbar$, where $m_P = (\hbar c / G)^{1/2}$, and $\delta s \sim \lambda_c \sim \hbar / mc$, with λ_c the Compton length, it becomes

$$\Delta x \Delta p \geq \frac{\hbar}{2} + \frac{\alpha}{c^3} G \Delta p^2, \quad (17)$$

that is we recover Eq. (1). α is a free parameter. Eq. (17) is the result which we want: the geometrical interpretation of QM through a quantization model formulated in a eight-dimensional manifold, implying the existence of an upper limit on the acceleration of particles, leads to the generalized principle of string theory.

It is worthwhile to note that, in the last term of (17), the dependence on \hbar disappears. So that this term is not related to quantum fluctuations but, as the uncertainty principle for strings, it is due to the intrinsic extension of particles.

4 Conclusions

Starting from the uncertainty principle of QM written in a space-time, where the effective geometry is induced by the acceleration of particles moving along their worldlines, the generalized uncertainty principle of string theory has been derived.

In this model we have assumed the maximal acceleration as an universal constant expressed in term of the Planck mass, which value is $\mathcal{A} \sim 10^{52} m / \text{sec}^2$. As expected,

it becomes relevant at very high energy where the emergence of a minimal observable distance occurs.

Unlike the string theory, in which the extension of particles is introduced *ab initio*, in quantum geometry such an extension is taken into account through the constraint of the maximal acceleration, that is by modifying the geometry in which moves an accelerating particle.

In this sense, we can state that the geometrical formulation of QM is an alternative approach in order to study physics of extended objects.

However, we have to note the fact that we have not used any second quantization scheme or full QFT approach in deriving our generalized uncertainty principle, nevertheless it is indicative of the fact that quantum geometry is an alternative scheme leading to physical interesting results.

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